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UIC

Mathématiques Appliquées

TC_ING

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Exemple de corrigé du Contrôle Continu

Exercice 1 :

Soit y solution de : $\boxed{x^2 y' = e^{-y}}$

si $x \neq 0$ (Condition (*)), ceci équivaut à : $y' e^{y} = \frac{1}{x^2}$

$$\iff (e^{y})' = \left(-\frac{1}{x}\right)'$$

$$\iff e^{y} = -\frac{1}{x} + c \quad , c \in \mathbb{R}$$

$$\iff \boxed{y = \ln\left(-\frac{1}{x} + c\right)} \text{ avec } -\frac{1}{x} + c > 0$$

$$-\frac{1}{x} + c > 0 \iff -\frac{1}{x} > -c$$

$$\iff \boxed{+\frac{1}{x} < c}$$

⚠ aux signes lors des multiplications / divisions

si $c > 0$

$\iff \begin{cases} x > \frac{1}{c} , \text{ si } x > 0 \\ \text{ou} \\ x \in \mathbb{R}^*_- , \text{ si } x < 0 \end{cases}$

$\iff x \in \mathbb{R}^*_- \cup \left]\frac{1}{c}, +\infty\right[$

(condition (*) vérifiée)

si $c = 0$

$\iff \frac{1}{x} < 0$

$\iff x < 0$

$\iff x \in \mathbb{R}^*_-$

(condition (*) vérifiée)

si $c < 0$

$\iff \frac{1}{x} < 0$ et $\frac{1}{x} < c$

$\iff x < 0$ et $x > \frac{1}{c}$

$\iff \frac{1}{c} < x < 0$

$\iff x \in \left]\frac{1}{c}, 0\right[$

(condition (*) vérifiée)

Conclusion : les solutions de $x^2 y' = e^{-y}$ sont :

$$y = \ln\left(-\frac{1}{x} + c\right)$$

avec $I = \mathbb{R}^*_- \cup \left]\frac{1}{c}, +\infty\right[$ si $c > 0$

$I = \mathbb{R}^*_-$ si $c = 0$

$I = \left]\frac{1}{c}, 0\right[$ si $c < 0$

I étant le domaine d'appartenance de x.

Exercice 2 :

$$\Pi(t) = \begin{cases} 1 & \text{si } t \in \left[-\frac{1}{2}, \frac{1}{2}\right) \\ 0 & \text{sinon} \end{cases}$$

a) soit $f : t \longmapsto \Pi\left(\frac{t-1}{2}\right)$ et $g : t \longmapsto \Pi\left(t - \frac{1}{2}\right)$

alors $f(t) = \Pi\left(\frac{t}{2} - \frac{1}{2}\right) = g\left(\frac{t}{2}\right)$

d'où $\mathcal{F}f(u) = \mathcal{F}\left(g\left(\frac{t}{2}\right)\right)(u) = \frac{1}{\left|\frac{1}{2}\right|} \cdot \mathcal{F}g\left(\frac{u}{\frac{1}{2}}\right) = 2\,\mathcal{F}g(2u)$

et $\mathcal{F}g(u) = \mathcal{F}\left(\Pi\left(t - \frac{1}{2}\right)\right)(u) = e^{-2i\pi \cdot \frac{1}{2} \cdot u} \cdot \mathcal{F}\Pi(u)$

$= e^{-i\pi u} \cdot \mathcal{F}\Pi(u)$

$\mathcal{F}g(u) = e^{-i\pi u} \cdot \text{sinc}(\pi u)$

d'où $\mathcal{F}f(u) = 2 \cdot \mathcal{F}g(2u) = 2 \cdot \left(e^{-i\pi(2u)} \cdot \text{sinc}(\pi \cdot 2u)\right)$

d'où $\boxed{\mathcal{F}\left(\Pi\left(\frac{t-1}{2}\right)\right)(u) = 2 \cdot e^{-2i\pi u}\,\text{sinc}(2\pi u)}$

b) $\mathcal{F}\left(t.\Pi(t)\right)(u) = \left(\frac{i}{2\pi}\right) \frac{d\,\mathcal{F}\Pi(u)}{du}$

$= \frac{i}{2\pi} \cdot \frac{d}{du}\left(\text{sinc}(\pi u)\right)$

$= \frac{i}{2\pi} \cdot \pi \cdot \text{sinc}'(\pi u)$

$\text{sinc}'(u) = \left(\frac{\sin(u)}{u}\right)' = \frac{\cos(u).u - \sin u}{u^2}$ (1)

d'où $\boxed{\mathcal{F}\left(t.\Pi(t)\right)(u) = \frac{i\left(\pi u.\cos(\pi u) - \sin(\pi u)\right)}{2\pi^2 u^2}}$

c) $\mathcal{F}\left(t^2.\Pi\left(\frac{t}{a}\right)\right)(u)$, $a \neq 0$

On a $\mathcal{F}\left(\Pi\left(\frac{t}{a}\right)\right)(u) = \frac{1}{\left|\frac{1}{a}\right|} \cdot \mathcal{F}\Pi\left(\frac{u}{\frac{1}{a}}\right)$

$= |a| \cdot \mathcal{F}\Pi(a u)$

$= |a| \cdot \text{sinc}(\pi a u)$

et $\mathcal{F}\left(t^2.\Pi\left(\frac{t}{a}\right)\right)(u) = \left(\frac{i}{2\pi}\right)^2 \cdot \frac{d^2}{du^2}\left(\mathcal{F}\left(\Pi\left(\frac{t}{a}\right)\right)(u)\right)$

d'où $\mathcal{F}\left(t^2.\Pi\left(\frac{t}{a}\right)\right)(u) = -\frac{1}{4\pi^2} \frac{d^2}{du^2}\left(|a|.\text{sinc}(\pi a u)\right)$

$\left(\text{sinc}(\pi a u)\right)' = \left(\frac{\sin(\pi a u)}{\pi a u}\right)' = \frac{\pi a\cos(\pi a u).\pi a u - \pi a\sin(\pi a u)}{(\pi a u)^2}$

on $\left(\text{sinc}(\pi a u)\right)' = \pi a.\text{sinc}'(\pi a u) \overset{(1)}{=} \pi a \cdot \frac{\cos(\pi a u).\pi a u - \sin(\pi a u)}{(\pi a u)^2}$

et $\left(\sin(\pi a u)\right)'' = \pi a.\pi a.\text{sinc}''(\pi a u) = (\pi a)^2 \cdot \text{sinc}''(\pi a u)$

et $\text{sinc}''(u) = \left(\frac{\cos(u).u - \sin u}{u^2}\right)' = \frac{(-\sin u.u + \cos u - \cos u)u^2 - 2u(\cos(u).u - \sin u)}{u^4}$

$$= \frac{-u^3 \cdot \sin u - 2u^2 \cos u + 2u \sin u}{u^4}$$

$$\text{sinc}''(u) = \frac{-u^2 \sin u - 2u \cos u + 2 \sin u}{u^3}$$

d'où : $\mathcal{F}\left(t^2 \cdot \Pi\left(\frac{t}{a}\right)\right)(u) = -\frac{|a|}{4\pi^2} \cdot (\pi a)^2 \cdot \frac{-(\pi a u)^2 \sin(\pi a u) - 2\pi a u \cos(\pi a u) + 2 \sin(\pi a u)}{(\pi a u)^3}$

$$\boxed{\mathcal{F}\left(t^2 \cdot \Pi\left(\frac{t}{a}\right)\right)(u) = |a| \cdot \frac{(\pi a u)^2 \sin(\pi a u) + 2\pi a u \cdot \cos(\pi a u) - 2 \sin(\pi a u)}{4\pi^3 a u^3}}$$

Exercice 3 :

$$\alpha > 0, \quad \mathcal{F}\left(e^{-\alpha t^2}\right)(u) = \sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{\pi^2}{\alpha} \cdot u^2}$$

$a > 0$

$\forall x \in \mathbb{R} \quad f_a(x) = \frac{1}{a} e^{-a^2 x^2}$

On a : $\mathcal{F}(f_a * f_b)(u) = \mathcal{F} f_a(u) \cdot \mathcal{F} f_b(u)$

et $\mathcal{F} f_a(u) = \frac{1}{a} \cdot \sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{\pi^2}{\alpha} u^2}$ avec $\alpha = a^2$

$$= \frac{1}{a} \sqrt{\frac{\pi}{a^2}} \cdot e^{-\frac{\pi^2}{a^2} u^2}$$

$$= \frac{\sqrt{\pi}}{a^2} \cdot e^{-\frac{\pi^2 u^2}{a^2}}$$

de même $\mathcal{F} f_b(u) = \frac{\sqrt{\pi}}{b^2} \cdot e^{\frac{-\pi^2 u^2}{b^2}}$

d'où : $\mathcal{F}(f_a * f_b)(u) = \frac{\pi}{a^2 \cdot b^2} \, e^{-\pi^2 u^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right)}$

Posons :

$$\frac{1}{A^2} = \frac{1}{a^2} + \frac{1}{b^2} = \frac{a^2+b^2}{a^2.b^2} \Rightarrow A = \frac{a.b}{\sqrt{a^2+b^2}}$$

et

$$\mathcal{F} f_A(\mu) = \frac{\sqrt{\pi}}{A^2} \cdot e^{-\frac{\pi^2\mu^2}{A^2}} = \frac{\sqrt{\pi}}{a^2.b^2}(a^2+b^2) \cdot e^{-\pi^2\mu^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$$

$$= \frac{a^2+b^2}{\sqrt{\pi}} \cdot \frac{\pi}{a^2.b^2} e^{-\pi^2\mu^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$$

$$= \frac{a^2+b^2}{\sqrt{\pi}} \cdot \mathcal{F}(f_a * f_b)(\mu)$$

d'où

$$\mathcal{F}(f_a * f_b)(\mu) = \mathcal{F}\left(\frac{\sqrt{\pi}}{a^2+b^2} f_A\right)(\mu)$$ (par linéarité de la TF)

d'où

$$\boxed{f_a * f_b = \frac{\sqrt{\pi}}{a^2+b^2} f_{\frac{a.b}{\sqrt{a^2+b^2}}}}$$

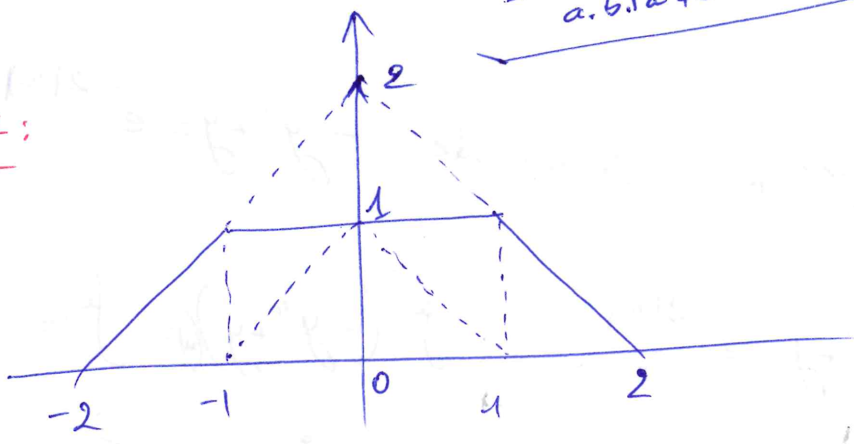
d'où $\forall x \in \mathbb{R}$,

$$(f_a * f_b)(x) = \frac{\sqrt{\pi}}{a^2+b^2} \frac{1}{\frac{a.b}{\sqrt{a^2+b^2}}} e^{-\frac{a^2.b^2}{a^2+b^2}x^2}$$

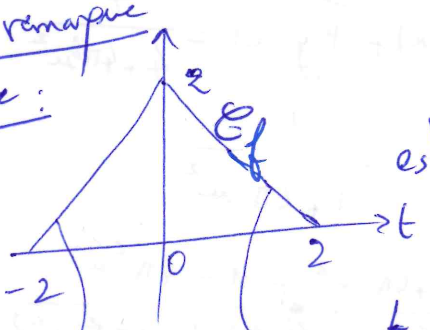
$$\boxed{= \frac{\sqrt{\pi}}{a.b\sqrt{a^2+b^2}} \cdot e^{-\frac{a^2.b^2}{a^2+b^2}x^2}}$$

Exercice 4 :

a)



On remarque que :



est le graphe de la fonction $\boxed{t \mapsto 2.\Lambda\left(\frac{t}{2}\right) = f(t)}$

$\to -t+b$ avec $-2+b=0 \Rightarrow b=2$

$\Rightarrow -t+2 = 2-t = 2\left(1-\frac{t}{2}\right) = 2\Lambda\left(\frac{t}{2}\right)$ sur $\mathbb{R}^+$

$t+2$

Vérification : $f(0) = 2\Lambda(0) = 2\times1 = 2$; $f(2) = 2\Lambda(1) = 2.0 = 0$

et on remarque que $g = f - \Lambda$ (graphiquement)

d'où $\boxed{g(t) = 2\Lambda\left(\frac{t}{2}\right) - \Lambda(t), \quad \forall t \in \mathbb{R}}$

b) D'après a),

$$\mathcal{F}g(u) = 2\mathcal{F}\left(\Lambda\left(\frac{t}{2}\right)\right)(u) - \mathcal{F}\Lambda(u)$$

$$= 2 \cdot \frac{1}{\left|\frac{1}{2}\right|} \mathcal{F}\Lambda\left(\frac{u}{\frac{1}{2}}\right) - \operatorname{sinc}^2(\pi u)$$

$$= 4 \cdot \mathcal{F}\Lambda(2u) - \operatorname{sinc}^2(\pi u)$$

$$\boxed{\mathcal{F}g(u) = 4 \cdot \operatorname{sinc}^2(2\pi u) - \operatorname{sinc}^2(\pi u) = \frac{\sin^2(2\pi u) - \sin^2(\pi u)}{\pi^2 u^2}}$$

Exercice 5 :

$f \in \mathcal{L}^1(\mathbb{R})$ solution de $-y'' + y = e^{-2|x|}$

a)
$$-y'' + y = e^{-2|x|} \Rightarrow \mathcal{F}(-y'' + y)(u) = \mathcal{F}\left(e^{-2|x|}\right)(u)$$

$$\Rightarrow -(2i\pi u)^2 \mathcal{F}y(u) + \mathcal{F}y(u) = \frac{4}{4 + 4\pi^2 u^2}$$

$$\Rightarrow \mathcal{F}y(u) = \frac{1}{1 + 4\pi^2 u^2} \cdot \frac{1}{1 + 4\pi^2 u^2}$$

et on a
$$\frac{4}{3}\left(\frac{1}{1 + 4\pi^2 u^2} - \frac{1}{4 + 4\pi^2 u^2}\right) = \frac{4}{3} \cdot \frac{4 + \cancel{4\pi^2 u^2} - 1 - \cancel{4\pi^2 u^2}}{(1 + 4\pi^2 u^2) \cdot (4 + 4\pi^2 u^2)}$$

$$= \frac{1}{(1 + 4\pi^2 u^2) \cdot (1 + 4\pi^2 u^2)}$$

d'où on a bien :

$$\boxed{\mathcal{F}f(\nu) = \frac{4}{3}\left(\frac{1}{1+4\pi^2\nu^2} - \frac{1}{4+4\pi^2\nu^2}\right)}$$

b) On sait que $\mathcal{F}\left(e^{-a|t|}\right)(\nu) \overset{a>0}{=} \frac{2a}{a^2+4\pi^2\nu^2}$

d'où $\mathcal{F}f(\nu) = \frac{4}{3}\,\mathcal{F}\left(\frac{1}{2}\cdot e^{-|x|} - \frac{1}{4}e^{-2|x|}\right)(\nu)$

d'où $\boxed{f(x) = \frac{2}{3}e^{-|x|} - \frac{1}{3}e^{-2|x|}}$ sur $\mathbb{R}$.