UIC | Pr. Morad Lakhssassi | (1

Corrigé Contrôle C. N°1 — ANALYSE 2 — CPI 1

Exercice 1 :

a) $F(t)=\int^t (u^3-u^2+u).\cos(u)\,du = \left[(u^3-u^2+u)\sin u\right]^t - \int (3u^2-2u+1)\sin u\,du$

$= (t^3-t^2+t)\sin t + cte + \left[(3u^2-2u+1)\cos u\right]^t$

$- \int^t (6u-2)\cos u\,du$

$= (t^3-t^2+t)\sin t + (3t^2-2t+1)\cos t - \left[(6u-2)\sin u\right]^t$

$+ \int^t 6.\sin u\,du + cte'$

$= (t^3-t^2+t)\sin t + (3t^2-2t+1)\cos t - (6t-2)\sin t$

$- 6\cos u + cte''$

$$\boxed{\int^t (u^3-u^2+u)\cos u\,du = (t^3-t^2-5t+2)\sin t + (3t^2-2t-5)\cos t + C}$$

b) $F(t)=\int^t \arcsin(u)\,du = \left[t\arcsin u\right]^t - \int^t u.\frac{1}{\sqrt{1-u^2}}\,du$

$= t\arcsin t + cte + \left[\sqrt{1-u^2}\right]^t$

$$\boxed{\int^t \arcsin(u)\,du = t\arcsin t + \sqrt{1-t^2} + C}$$

c) $F(t)=\int^t u^2.\ln u\,du = \left[\frac{u^3}{3}.\ln u\right]^t - \int^t \frac{u^3}{3}.\frac{1}{u}\,du$

$= \frac{t^3\ln t}{3} + cte - \int^t \frac{u^2}{3}\,du = \frac{t^3\ln t}{3} - \frac{t^3}{9} + C$

d) $I = \int^t \cos u . e^u \, du = \left[e^u . \cos u\right]^t + \int^t e^u \sin u \, du$

$= e^t . \cos t + cte + \left[e^u . \sin u\right]^t - \int^t e^u \cos u \, du$

$= e^t (\cos t + \sin t) + C - I$

$\Rightarrow \boxed{I = \frac{e^t}{2} (\cos t + \sin t) + C'}$

Exercice 2 :

a) $\boxed{I = \int_1^e \frac{dt}{t\sqrt{\ln(t)+1}}}$:

$x := \ln t \Rightarrow dx = \frac{1}{t} dt$

$t = 1 \Rightarrow x = 0, \quad t = e \Rightarrow x = 1$

$\Rightarrow I = \int_0^1 \frac{dx}{\sqrt{x+1}} = 2\left[\sqrt{x+1}\right]_0^1 = \boxed{2\sqrt{2} - 2}$.

b) $I = \boxed{\int_0^1 \frac{dt}{e^t+1}}$

$x := e^t \Rightarrow dx = e^t dt = x \, dt \Rightarrow dt = \frac{dx}{x}$

$t = 0 \Rightarrow x = 1, \quad t = 1 \Rightarrow x = e$

$\Rightarrow I = \boxed{\int_1^e \frac{dx}{x(x+1)}} = \int_1^e \frac{1}{x} - \frac{1}{x+1} \, dx$

$= \left[\ln x - \ln(x+1)\right]_1^e$

$= 1 - \ln(e+1) - (0 - \ln 2)$

$= \boxed{1 - \ln(e+1) + \ln 2}$

Exercice 3 :

a) $I = \int_0^1 \frac{1}{(1+x^2)^2}\,dx$ avec le chgt de variable $x := \tan(t)$

$x := \tan(t) \Rightarrow dx = \frac{1}{\cos^2 t}\,dt$

$x = 0 \Rightarrow t = \arctan(0) = 0$

$x = 1 \Rightarrow t = \pi/4$

$1 + x^2 = 1 + \tan^2 t = \frac{1}{\cos^2 t}$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\cos^4 t}{\cos^2 t}\,dt = \int_0^{\frac{\pi}{4}} \cos^2 t\,dt$$

ou

$x := \tan t \Rightarrow t = \arctan(x)$

$\Rightarrow dt = \frac{1}{1+x^2}\,dx$

$\Rightarrow \frac{1}{(1+x^2)^2}\,dx = \frac{1}{1+x^2}\,dt = \frac{1}{1+\tan^2 t}\,dt$

$= \cos^2 t\,dt$

$$= \int_0^{\pi/4} \frac{1}{2}(\cos 2t + 1)\,dt$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{1}{2}(\sin 2t)' + t'\,dt$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 2t + t\right]_0^{\pi/4} = \frac{1}{2}\left(\frac{1}{2} + \frac{\pi}{4} - 0\right)$$

$$= \frac{1}{4} + \frac{\pi}{8}$$

$$\Rightarrow \boxed{\int_0^1 \frac{1}{(1+x^2)^2}\,dx = \frac{1}{4} + \frac{\pi}{8}}$$

b) $J = \int_1^2 \frac{\ln(1+t) - \ln(t)}{t^2}\,dt$ avec le chgt de variable $x := \frac{1}{t}$

$x := \frac{1}{t} \Rightarrow t = \frac{1}{x} \Rightarrow dt = -\frac{1}{x^2}\,dx$ ou $dx = -\frac{1}{t^2}\,dt$

$t = 1 \Rightarrow x = 1$, $t = 2 \Rightarrow x = \frac{1}{2}$

$$J = \int_1^2 \frac{\ln\left(1 + \frac{1}{x}\right) - \ln\left(\frac{1}{x}\right)}{1} \times \frac{dt}{t^2}$$

$$= \int_1^{1/2} \ln\left(1 + \frac{1}{x}\right) - \ln\left(\frac{1}{x}\right) \times (-dx)$$

$$= -\int_1^{1/2} \ln\left(\frac{1+\frac{1}{x}}{\frac{1}{x}}\right)dx = +\int_{\frac{1}{2}}^{1} \ln(1+x)\,dx$$

$u := 1 + x \Rightarrow du = dx$

Exercice 4 :

a) $\boxed{\sum_{k=0}^{n} \frac{1}{k+n}} = \frac{1}{n} + \sum_{k=1}^{n} \frac{1}{n(k/n+1)} = \frac{1}{n} + \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+k/n}$

$\boxed{\xrightarrow[n\to+\infty]{}} 0 + \int_0^1 \frac{1}{1+x}\,dx$

$= \left[\ln(1+x)\right]_0^1$

$= \boxed{\ln 2}$

b) $\boxed{\sum_{k=1}^{n} \sin\left(\frac{k}{n}\right)\cdot\frac{k}{n^2}} = \frac{1}{n}\sum_{k=1}^{n} \sin\left(\frac{k}{n}\right)\cdot\frac{k}{n}$

$\boxed{\xrightarrow[n\to+\infty]{}} \int_0^1 x\cdot\sin x\,dx$

$= \left[-x.\cos x\right]_0^1 + \int_0^1 \cos x\,dx$

$= -\cos(1) + \left(\sin x\right)_0^1$

$= -\cos 1 + \sin 1$

$\boxed{= \sin 1 - \cos 1}$

Exercice 5 :

a) $\forall n \in \mathbb{N}$, $I_n = \int_0^{\pi/2} (\sin(x))^n\,dx$

$I_{n+2} = \int_0^{\pi/2} \sin^{n+2}(x)\,dx = \int_0^{\pi/2} \sin^{n+1}(x).\sin(x)\,dx$

$= \left[\sin^{n+1}(x).(-\cos(x))\,dx\right]_0^{\pi/2} + \int_0 (n+1)\cos(x).\sin^n(x).\cos(x)\,dx$

$= 0 + (n+1)\int^{\pi/2} \cos^2(x).\sin^n(x)\,dx$

$$\Rightarrow I_{n+2} = (n+1)\left(I_n - I_{n+2}\right)$$

$$\Rightarrow I_{n+2}(1+n+1) = (n+1) I_n$$

$$\Rightarrow \boxed{I_{n+2} = \frac{n+1}{n+2} I_n}, \forall n \in \mathbb{N}$$

b) $\forall n \in \mathbb{N}$, $$I_{n+2} = \frac{n+1}{n+2} I_n = \frac{n+1}{n+2} \cdot \frac{n-1}{n} I_{n-2}$$

$$= \frac{n+1}{n+2} \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \times \cdots \times \begin{cases} \frac{1}{2} I_0 \\ \text{ou} \\ \frac{2}{3} I_1 \end{cases}$$ selon la parité de n

$\boxed{\text{Si } n \text{ pair}}$ $$I_{2p} = \frac{2p-1}{2p} I_{2p-2}$$

$$= \frac{2p-1}{2p} \cdot \frac{2p-3}{2p-2} \cdot \frac{2p-5}{2p-4} \times \cdots \times \frac{3}{4} \times \frac{1}{2} I_0$$

$$= \frac{2p \cdot (2p-1)(2p-2)(2p-3) \times \cdots \times 3 \times 2 \times 1}{(2p)^2 (2p-2)^2 (2p-4)^2 \times \cdots \times 4^2 \times 2^2} I_0$$

$$= \frac{(2p)!}{(4)^p \, p(p-1)(p-2) \times \cdots \times 2 \times 1} I_0$$

$$I_{2p} = \frac{(2p)!}{4^p \cdot p!} I_0$$

avec $I_0 = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2}$

$$\Rightarrow \boxed{I_{2p} = \frac{(2p)!}{4^p \cdot p!} \frac{\pi}{2}}$$

$\boxed{\text{Si } n \text{ impair}}$ $$I_{2p+1} = \frac{2p}{2p+1} I_{2p-1}$$

$$= \frac{2p}{2p+1} \cdot \frac{2p-2}{2p-1} \cdot \frac{2p-4}{2p-3} \times \cdots \times \frac{4}{5} \times \frac{2}{3} I_1$$

$$= \frac{(2p)^2 \cdot (2p-2)^2 \cdot (2p-4)^2 \times \cdots \times 4^2 \times 2^2}{(2p+1)(2p)(2p-1)(2p-2) \times \cdots \times 5 \times 4 \times 3 \times 2} I_1$$

$$I_1 = \int_0^{\pi/2} \sin(x)\, dx$$

$$= \left(-\cos x\right]_0^{\pi/2} = +1$$

$$\Rightarrow \quad \boxed{I_{2p+1} = + \frac{4^p \cdot p!}{(2p+1)!}}$$