CORRIGÉ DU CONTROLE N°2

Exercice 1

1/ On rappelle que $u'(x) \simeq \dfrac{u(x+h)-u(x)}{h}$

$$u''(x) \simeq \frac{u(x+h)-2u(x)+u(x-h)}{h^2}$$

En remplaçant dans le problème (P), on obtient

$$\begin{cases} -\dfrac{u_{i+1}-2u_i+u_{i-1}}{h^2} + (1+ih)\dfrac{u_{i+1}-u_i}{h} + ih\,u_i = \cos(ih) & 1 \le i \le N-1 \\ u_0 = 1, \quad u_N = -1 \end{cases}$$

(2) Cet algorithme s'écrit :

$$\begin{cases} -\dfrac{1}{h^2}u_{i-1} + \left(\dfrac{2}{h^2} - \dfrac{(1+ih)}{h} + ih\right)u_i + \left(-\dfrac{1}{h^2} + \dfrac{1+ih}{h}\right)u_{i+1} = \cos(ih) & i \in [\![1, N-1]\!] \\ u_0 = 1, \quad u_N = 1 \end{cases}$$

posons $\alpha_i = -1/h^2$, $\beta_i = \dfrac{2}{h^2} - \dfrac{1+ih}{h} + ih$, $\gamma_i = -\dfrac{1}{h^2} + \dfrac{1+ih}{h}$

$$\begin{cases} \alpha_i u_{i-1} + \beta_i u_i + \gamma_i u_{i+1} = \cos(ih) & 1 \le i \le N-1 \\ u_0 = 1, \quad u_N = -1 \end{cases}$$

$i = 1$

$$\beta_1 U_1 + \gamma_1 U_2 = \cos(h) - \alpha_1$$

$i = 2$

$$\alpha_2 U_1 + \beta_2 U_2 + \gamma_2 U_3 = \cos(2h)$$

$\vdots$

$i = N-1$

$$\alpha_{N-1} U_{N-2} + \beta_{N-1} U_{N-1} = \cos((N-1)h) + \gamma_{N-1}$$

c'est à dire

$$\underbrace{\begin{pmatrix} \beta_1 & \gamma_1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & 0 \\ & 0 & & & \alpha_{N-1} & \beta_{N-1} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} U_1 \\ \vdots \\ \vdots \\ U_{N-1} \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} \cos h - \alpha_1 \\ \cos(2h) \\ \vdots \\ \cos(N-2)h \\ \cos(N-1)h + \gamma_{N-1} \end{pmatrix}}_{B}$$

Exercice 2

On rappelle que :

$$\frac{\partial u}{\partial t}(x,t) \simeq \frac{u(x,t+k)-u(x,t)}{k}$$

$$\frac{\partial^2 u}{\partial x^2}(x,t) \simeq \frac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2}$$

1/ Schéma Explicite

$$\begin{cases} \dfrac{u_{ij}-u_{ij-1}}{k} - \dfrac{u_{i+1\,j-1}-2u_{ij-1}+u_{i-1\,j-1}}{h^2} = e^{-t_{j-1}}\cos(x_i) & 1 \le i \le N,\ 1 \le j \le M+1 \\ (1)\ u_{0j} = t_j e^{-t_j} & \\ (2)\ u_{N+1\,j} = e^{-t_j} & \Big\}\ j \in [\![0, M+1]\!] \\ (3)\ u_{i0} = 1+\sin(x_i) & i \in [\![0, N+1]\!] \end{cases}$$

2/ Schéma Implicite

$$\begin{cases} \dfrac{u_{ij}-u_{ij-1}}{k} - \dfrac{u_{i+1\,j}-2u_{ij}+u_{i-1\,j}}{h^2} = e^{-t_j}\cos(x_i) & 1 \le i \le N,\ 1 \le j \le M+1 \\ (1)+(2)+(3) & \end{cases}$$

3/ Schéma de CRANK–NICOLSON

$$\begin{cases} \dfrac{u_{ij} - u_{ij-1}}{k} - \dfrac{1}{2}\left[\dfrac{u_{i+1\,j-1} - 2u_{ij-1} + u_{i-1\,j-1}}{h^2}\right] - \dfrac{1}{2}\left[\dfrac{u_{i+1\,j} - 2u_{ij} + u_{i-1\,j}}{h^2}\right] \\ \qquad = e^{-t_j}\cos(x_i) \qquad 1 \leq i \leq N \\ \qquad\qquad\qquad\qquad\quad 1 \leq j \leq M+1 \\ (1) + (2) + (3), \end{cases}$$