Mécanique des fluides avancée : corrigé TD1

Exercice 1 :

$$V_x = \frac{x}{t+t_0}$$

1/

$$div\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$
$$\Rightarrow \; div\vec{V} = \frac{1}{t+t_0} \neq 0$$

Ecoulement compressible.

2/

$$\vec{V} = V_x\vec{e}_x \quad ; \quad V_x = \frac{x}{t+t_0} \quad et \quad \frac{dx}{dt} = V_x$$
$$\frac{dx}{dt} = \frac{x}{t+t_0} \quad \Rightarrow \quad \frac{dx}{x} = \frac{dt}{t+t_0}$$
$$\ln(x) = ln(t+t_0) + C$$
$$\Rightarrow \quad x = e^c(t+t_0)$$

3/ équation de continuité :

$$\frac{\partial \rho}{\partial t} + div\rho\vec{V} = 0$$

$$\frac{\partial \rho}{\partial t} + \vec{V}\,\overrightarrow{grad}\,\rho + \rho\, div\vec{V} = 0$$
$$\frac{\partial \rho}{\partial t} + V_x\frac{\partial \rho}{\partial x} + \rho\frac{\partial V_x}{\partial x} = 0$$
$$V_x = \frac{x}{t+t_0} \quad \Rightarrow \frac{\partial \rho}{\partial t} + \frac{x}{t+t_0} \times \frac{\partial \rho}{\partial x} + \rho\frac{\partial V_x}{\partial x} = 0$$
$$\frac{\partial \rho}{\partial t} + \frac{x}{t+t_0} \times \frac{\partial \rho}{\partial x} + \frac{\partial \rho}{t+t_0} = 0$$
$$\frac{\partial \rho}{\partial t} + k\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{t+t_0} = 0$$
$$\frac{\partial \rho}{\partial t} + k\frac{\partial \rho}{\partial x} = -\frac{\partial \rho}{t+t_0}$$
$$\partial\rho = \frac{\partial \rho}{\partial x}\,\partial x + \frac{\partial \rho}{\partial t}\partial t$$
$$x = k(t+t_0) \quad \Rightarrow dx = kdt$$
$$\partial\rho = k\frac{\partial \rho}{\partial x}\,\partial t + \frac{\partial \rho}{\partial t}\partial t = \left(k\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial t}\right)\partial t$$
$$\Rightarrow \partial\rho = \frac{-\rho}{t+t_0}dt$$

$$\Longrightarrow ln\rho = ln\frac{1}{t+t_0} + lnt_0$$

$$\Longrightarrow \frac{\rho}{\rho_0} = \frac{t_0}{t+t_0} \quad \Longrightarrow \quad \rho = \frac{t_0.\rho(x,y,z,0)}{t+t_0}$$

Exercice 2 :

1/ Equation de NS :

$$(A) : \begin{cases} -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 U}{\partial z^2} = \rho a_x \\ -\frac{\partial p}{\partial y} = \rho a_y \\ -\frac{\partial p}{\partial z} = \rho a_z \end{cases}$$

$$\vec{a} = \frac{\partial \vec{V}}{\partial t} + (\vec{V}.\overrightarrow{grad})\vec{V}$$

$$\vec{V}\begin{pmatrix} U(z) \\ 0 \\ 0 \end{pmatrix} \quad ; \quad \overrightarrow{grad} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\vec{V}.\overrightarrow{grad} = \begin{pmatrix} U(z) & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} U(z)\frac{\partial}{\partial x} \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \quad \vec{a} = (\vec{V}.\overrightarrow{grad})\vec{V} = \begin{pmatrix} U(z)\frac{\partial U(z)}{\partial x} \\ 0 \\ 0 \end{pmatrix} = \vec{0}$$

$$(A)\ devient \ : \begin{cases} -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 U}{\partial z^2} = 0 \\ -\frac{\partial p}{\partial y} = 0 \\ -\frac{\partial p}{\partial z} = 0 \end{cases}$$

2/

a/ L'expression de U en fonction de z :

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 U}{\partial z^2} = k$$
$$\mu \frac{\partial^2 U}{\partial z^2} = k \implies \mu \frac{\partial U}{\partial z} = kz + k'$$
$$\implies U(z) = \frac{k}{2} z^2 + k'z + \alpha$$
$$U(-e) = 0 \implies \frac{k}{2} e^2 - k'e + \alpha = 0$$

$$U(e) = 0 \implies \frac{k}{2} e^2 + k'e + \alpha = 0$$
$$\implies k' = 0 \quad et \quad \alpha = -\frac{k}{2} e^2$$
$$or : \frac{\partial p}{\partial x} = \mu k \implies p = \mu k x + c$$
$$p(0) = p_e = c$$
$$p(L) = p_s = \mu k L + p_e \implies k = \frac{p_s - p_e}{\mu L}$$
$$U(z) = \frac{-1}{2\mu L}(p_e - p_s) z^2 + \frac{e^2}{2\mu L}(p_e - p_s)$$
$$\implies U(z) = \frac{p_e - p_s}{2\mu L}(e^2 - z^2)$$

b/ L'expression de la contrainte de cisaillement :

$$\sigma_{xy} = \mu\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right)$$
$$U(z) = \frac{p_e - p_s}{2\mu L}(e^2 - z^2)$$
$$\frac{\partial U(z)}{\partial z} = \frac{p_e - p_s}{\mu L} z$$
$$\sigma_{xy} = \frac{p_e - p_s}{L} z$$

c/ L'expression du débit par unité de largeur :

$$Q_v = \iint_S \vec{U}\, d\vec{S} = \int_0^y \int_{-e}^{e} \frac{p_e - p_s}{2\mu L}(e^2 - z^2)\, dz dy$$
$$\implies Q_v = y \int_{-e}^{e} \frac{p_e - p_s}{2\mu L}(e^2 - z^2)\, dz$$
$$\implies Q_v / y = \frac{p_e - p_s}{6\mu L}[z^3 - e^2]_{-e}^{e}$$

$$\Longrightarrow Q_v/y = \frac{p_e - p_s}{3\mu L} e^3$$

d/ L'expression de la vitesse moyenne du fluide entre les deux plaques :

Lorsque z = 0 :

$$U(z) = \frac{p_e - p_s}{2\mu L}(e^2 - z^2)$$

$$\Longrightarrow U(z=0) = \frac{p_e - p_s}{2\mu L}(e^2 - 0)$$

$$\Longrightarrow U_{moyenne} = \frac{p_e - p_s}{2\mu L} e^2$$

Exercice 3 :

1/ Equation de continuité :

$$div\, \vec{v} = 0$$

$$\frac{\partial V_x}{\partial x} = 0 \quad \Longrightarrow \quad V_x \quad ne\ depend\ pas\ de\ x.$$

2/ l'accélération :

$$a = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V}.\overrightarrow{grad})\vec{V}$$

L'écoulement est stationnaire : $\frac{\partial \vec{V}}{\partial t} = 0$

$$\vec{V}.\overrightarrow{grad} = \begin{pmatrix} V_x & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} V_x \frac{\partial}{\partial x} \\ 0 \\ 0 \end{pmatrix}$$

$$(\vec{V}.\overrightarrow{grad})\vec{V} = \begin{pmatrix} V_x \frac{\partial V_x}{\partial x} \\ 0 \\ 0 \end{pmatrix} = 0$$

$$donc : \quad \vec{a} = \vec{0}$$

3/ équation de Navier stocks :

$$\begin{cases} -\dfrac{\partial p}{\partial x} + \mu\dfrac{\partial^2 V_x}{\partial y^2} = 0 \\ -\rho g - \dfrac{\partial p}{\partial y} = 0 \\ -\dfrac{\partial p}{\partial z} = 0 \end{cases}$$

$$\vec{e}_y \ : \ \frac{\partial p}{\partial y} = -\rho g \quad \Rightarrow \ p = -\,\rho g y + k$$

$$p\left(\frac{e}{2}\right) = p_0 \quad \Rightarrow -\,\rho g\left(\frac{e}{2}\right) + k = p_0$$

$$k = p_0 + \rho g\left(\frac{e}{2}\right)$$

$$donc\ p = \ \rho g\left(\frac{e}{2} - y\right) + p_0$$

4/ L'expression de la vitesse :

$$\frac{\partial p}{\partial x} = 0 \quad \Rightarrow \ \mu\frac{\partial^2 V_x}{\partial y^2} = 0$$

$$\frac{\partial V_x}{\partial y} = k \quad \Rightarrow V_x(y) = ky + k'$$

5/

$$V_x\left(\frac{-e}{2}\right) = 0 \ \Rightarrow -\frac{ke}{2} + k' = 0$$

$$V_x\left(\frac{e}{2}\right) = U_0 \ \Rightarrow \frac{ke}{2} + k' = U_0$$

$$\Rightarrow k = \frac{U_0}{e} \quad et \ \ k' = \frac{U_0}{2}$$

$$\Rightarrow V_x(y) = \frac{U_0}{e}y + \frac{U_0}{2}$$

6/

$$S = e.l$$

$$D_v = \iint \vec{V}.d\vec{S} = \iint V_x(y)dzdy$$

$$\Rightarrow D_v = l.e.\frac{U_0}{2}$$

7/ La force appliquée par le couteau est la force opposée au cisaillement

$$\vec{F}_C = -\vec{F}_{Cis}$$

$$d\vec{F}_{Cis} = -\mu\frac{\partial V_x}{\partial y}dS \ \ \vec{e}_x$$

$$F_c = \iint \mu\frac{\partial V_x}{\partial y}dS \qquad avec\ dS = dxdz$$

$$F_c = \iint \mu \frac{U_0}{e} dV dz = \mu \frac{U_0}{e} L.e$$

$$\Longrightarrow F_c = \mu . U_0 . L$$